

## On the Non-existence of Limiting Lines in Transonic Flows

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We consider flows past airfoils which are irrotational and subsonic for low Mach numbers at large distances from the airfoil. If the Mach number at infinity is gradually increased, at some particular value a small supersonic region appears in the flow next to the airfoil. The flow appears to remain without shocks. If the Mach number at infinity is increased still more, a definite observable shock wave appears. The question arises as to why this shock wave appears.

It is possible and even quite probable, that no mixed flows without shocks exist as transition stages and that in general as soon as the supersonic region appears there is a shock wave which is at first so weak as to be unobserved.

However, if these continuous mixed flows with varying Mach number do exist, an explanation of why the continuous flow breaks down is needed, and it has been suggested by various authors that a "limiting line" appears in the flow. Tollmien, Ringleb, von Kármán, and Tsien have observed that, if flows and their corresponding profiles are constructed using solutions in the hodograph plane which depend continuously on some Mach number, there is always a critical Mach number where the mapping of the hodograph plane into the physical plane breaks down. In fact, the image in the physical plane has a fold in the supersonic region whose edge is known as the "limiting line." The Jacobian of the transformation from the hodograph plane to the physical plane vanishes along the limiting line.

Tsien [1] and von Kármán have proposed that continuous flow past a fixed airfoil also breaks down because of the appearance of a limiting line. Nikolskii and Taganov [2] have shown that such a limiting line would have to start on the sonic curve.

It was finally shown by Friedrichs [3] that limiting lines cannot appear anywhere in analytic flows which depend continuously on the entrance Mach number and that therefore the breakdown of potential flow must be due to other causes. (Friedrichs' proof was challenged in a controversial review by Tsien [4].) Manwell [5] has shortened the proof considerably and eliminated the difficult lemma supplied by Flanders in [3]. The proof presented here dispenses with the condition of analyticity and requires only continuous second derivatives of the stream function. In addition we shall show, as in [5], that in the construction of flows past continuously changing profiles, [6,7] etc., the incidence of a limiting line corresponds to a profile of infinite curvature.

These two results are formulated in two theorems.

**Theorem 1.** *A limiting line cannot appear in a plane continuous flow past an airfoil of bounded curvature if the flow depends continuously on the Mach number at infinity and has a bounded supersonic region.*

**Theorem 2.** *If a set of flows which depend continuously on a parameter is constructed in the hodograph plane together with the corresponding set of profiles and if, for some critical value of the parameter, a limiting line appears, then the corresponding profile has infinite curvature at some point.*

We make the following assumptions:<sup>1</sup>

(a) A solution in the hodograph plane of the equations of an isentropic, irrotational, plane flow is given; i.e., we have a potential function  $\phi(q, \theta)$  and a stream function  $\psi(q, \theta)$  which satisfy the equations

$$(1) \quad \begin{aligned} \phi_\theta &= \rho^{-1} q \psi, \\ \phi_q &= \rho^{-1} q^{-1} c^{-2} (q^2 - c^2) \psi, \end{aligned}$$

in a closed simply connected region  $D$  of the  $q, \theta$ -plane. Here  $\rho$  is a known function of  $q$  obtained from the Bernoulli theorem for steady isentropic irrotational flow, and  $c^2 = dp/d\rho$ , where  $p = p(\rho)$  is the equation of state relating pressure to density.

(b)  $\psi_{,\theta\theta}$ ,  $\psi_{,\theta q}$ ,  $\psi_{,qq}$  exist and are bounded in  $D$ .

(c) The boundary  $C$  of  $D$  intersects the sonic line  $q = c = c_s$  in exactly two points.

(d)  $\rho > 0$  throughout  $D$ .

(e) The equation of state  $p = p(\rho)$  is such that  $d^2 p/d\rho^2 > 0$ .

A solution  $\phi, \psi$  of equations (1) gives rise to a flow in the physical plane provided that the Jacobian  $J$  of the transformation

$$x = x(q, \theta)$$

$$y = y(q, \theta)$$

does not vanish.  $J$  is given by

$$(2) \quad \begin{aligned} J &= \frac{\partial(x, y)}{\partial(\theta, q)} = \frac{\partial(\psi, \phi)}{\partial(q, \theta)} \left( \frac{\partial(\phi, \psi)}{\partial(x, y)} \right)^{-1} = \frac{1}{\rho q^3} (\phi_\theta \psi_q - \psi_\theta \phi_q) \\ &= \frac{1}{\rho^{\frac{1}{2}} q} (\psi_q^2 - (q^2/c^2 - 1) q^{-2} \psi_\theta^2) = q^{-2} (\phi_\theta^2 - \rho^{-2} c^{-2} (q^2 - c^2) \psi_\theta^2). \end{aligned}$$

A limiting line occurs if  $J$  vanishes along a curve. This cannot happen in the subsonic region, see [2], [3], and [8]. Thus a limiting line can occur only in the supersonic region or on the sonic curve.

In connection with  $J$  we now prove the fundamental

<sup>1</sup>Standard notation for the flow quantities is used.

*Lemma:* If  $J > 0$  on  $C$  then  $J > 0$  in  $D$  for  $q \geq c_*$ .

*Proof:* We introduce the new dependent variables

$$\begin{aligned} U &= \phi_* - k\psi_*, \\ V &= \phi_* + k\psi_*, \end{aligned} \quad (3)$$

where

$$k^2 = \rho^{-2} c^{-2} (q^2 - c^2). \quad (4)$$

With this change of variables,  $J$  becomes, by (2)

$$J = q^{-2} UV; \quad (5)$$

hence  $J = 0$  if and only if  $U = 0$  or  $V = 0$  since by assumption (d) both  $U$  and  $V$  are bounded in  $D$ .

From the differential equations (1) we find differential equations for  $U$  and  $V$ . First we introduce the characteristic variables  $\alpha$  and  $\beta$  by the equations

$$\begin{aligned} 2 d\alpha &= \frac{\rho k}{q} d\tau - d\theta \\ 2 d\beta &= \frac{\rho k}{q} dq + d\theta. \end{aligned} \quad (6)$$

Then we have

$$\begin{aligned} \frac{\partial}{\partial \alpha} &= \frac{q}{\rho k} \frac{\partial}{\partial q} - \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \beta} &= \frac{q}{\rho k} \frac{\partial}{\partial q} + \frac{\partial}{\partial \theta} \end{aligned} \quad (7)$$

and from differentiating (3) we find, using (1),

$$\begin{aligned} U_\beta &= \rho^{-1} k^{-1} q (\phi_{*\beta} - k\psi_{*\beta} - k_\beta \psi_*) + \phi_{*\beta} - k\psi_{*\beta} \\ &= -\rho^{-1} k^{-1} q k_\beta \psi_*, \end{aligned}$$

or by (3)

$$U_\beta = \frac{1}{2} \rho^{-1} k^{-2} q k_\beta (U - V). \quad (8)$$

Similarly,

$$V_\alpha = \frac{1}{2} \rho^{-1} k^{-2} q k_\alpha (V - U). \quad (9)$$

But, by (7), since  $k$  is a function of  $q$  alone, we have

$$k_\alpha = \rho q^{-1} k k_\alpha = \rho q^{-1} k k_\beta.$$

Substituting in (8) and (9), we thus obtain the equations

$$(10) \quad U_s - \frac{1}{2}k^{-1}k_s U = -\frac{1}{2}\rho^{-1}k^{-2}qk_s V$$

$$V_s - \frac{1}{2}k^{-1}k_s V = -\frac{1}{2}\rho^{-1}k^{-2}qk_s U$$

which may be rewritten as

$$(11) \quad (k^{-1}U^2)_s = -BUV$$

$$(k^{-1}V^2)_s = -BUV$$

where, by (4), (10),

$$\begin{aligned} B &= \rho^{-1}k^{-2}qk_s = \frac{1}{2}\rho^{-1}k^{-2}q \frac{d}{dq} \left( \frac{q^2 - c^2}{\rho^{\frac{2}{3}}c^2} \right) \\ &= \frac{1}{2}\rho^{-1}k^{-2}q \left\{ \frac{-\rho_s}{\rho^{\frac{2}{3}}} \left( \frac{q^2}{c^2} - 1 \right) + \frac{2q}{\rho^{\frac{2}{3}}c^2} - \frac{q^2}{\rho^{\frac{2}{3}}c^4} \frac{dc^2}{d\rho} \rho_s \right\}. \end{aligned}$$

Thus  $B > 0$  for  $q > c_* > c$  since  $\rho_s < 0$  by Bernoulli's theorem and  $dc^2/d\rho = d^2p/d\rho^2 > 0$  by assumption (e).

Let us suppose first that  $J$  vanishes or is negative somewhere in the supersonic region  $q > c_* > c$  of  $D$ . Since  $J$  is continuous and  $D$  is closed, there is a maximum value of  $q > c_*$  for which  $J = 0$ . Hence by (6) and (4), there is a maximum value  $\tau = \alpha^* + \beta^*$  of  $\alpha + \beta$  for which  $J = 0$ , and, for  $\alpha + \beta > \tau$ ,

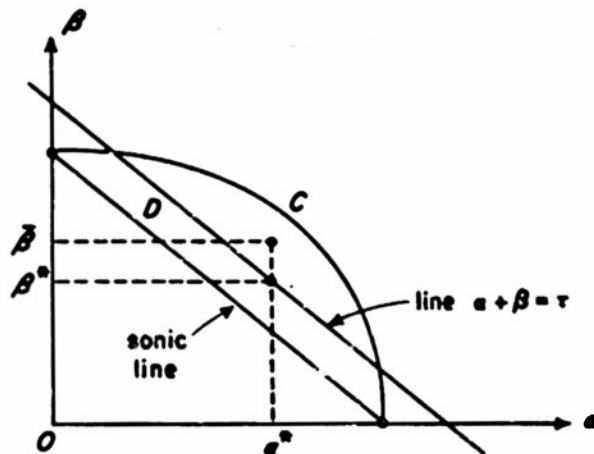


FIGURE 1

$J > 0$ , since  $J > 0$  on  $C$ . By (5), either  $U = 0$  or  $V = 0$  for  $\alpha = \alpha^*$ ,  $\beta = \beta^*$ . Suppose  $U = 0$  at this point. Then  $k^{-1}U^2 = 0$  at  $(\alpha^*, \beta^*)$  but  $k^{-1}U^2 > 0$  for  $\alpha + \beta > \tau$ . Hence, for  $\alpha = \alpha^*$ ,  $\beta > \beta^*$ ,  $k^{-1}U^2 > 0$  and therefore, by the mean value theorem,  $(k^{-1}U^2)_s > 0$  for  $\alpha = \alpha^*$  and some value  $\bar{\beta}$  of  $\beta$  where  $\bar{\beta} > \beta^*$ . But then by (11) since  $B > 0$  we have  $UV < 0$  at  $(\alpha^*, \bar{\beta})$ , that is, by (5),  $J < 0$  at  $(\alpha^*, \bar{\beta})$ . Since  $\alpha^* + \bar{\beta} > \tau$  this contradicts  $J > 0$  for  $\alpha + \beta > \tau$  and hence  $U$  cannot vanish for  $q > c_*$ . Similarly  $V$  cannot vanish, and thus  $J$  cannot vanish for  $q > c_* > c$ .

Next suppose that  $J = 0$  at a point  $P$  where  $q(P) = c_*$ , and let  $\alpha(P)$ ,  $\beta(P)$  be the coordinates of this point in the characteristic coordinate system. By (11), since  $J > 0$  and hence  $UV > 0$  for  $q > c_*$ ,  $k^{-1}U^2$  is decreasing in the direction of increasing  $\beta$  along the line  $\alpha = \alpha(P)$ . Since  $k^{-1}U^2$  is positive and continuous for  $q > c_*$ ,  $k^{-1}U^2$  must be bounded away from zero for  $q \geq c_*$ . Hence  $|k^{-1/2}U|$  is bounded away from zero on  $\alpha = \alpha(P)$ ,  $\beta \geq \beta(P)$ .

In terms of  $\psi_*$  we find from (3) that

$$(13) \quad k^{-1/2}\psi_* = \frac{p}{q}(k^{-1/2}U + k^{1/2}\psi_*)$$

and hence, since  $\psi_*$  is bounded and  $k \rightarrow 0$  for  $\alpha = \alpha(P)$ ,  $\beta \rightarrow \beta(P)$  from above,  $|k^{-1/2}\psi_*|$  is also bounded away from zero for  $\alpha = \alpha(P)$ ,  $\beta \geq \beta(P)$ ,  $\beta$  sufficiently close to  $\beta(P)$ . Therefore, using (4) expanded about  $q = c = c_*$ , we have

$$(14) \quad \left| \frac{\psi_*}{(q - c_*)^{1/4}} \right| > \delta > 0$$

for  $\alpha = \alpha(P)$ ,  $\beta \geq \beta(P)$ ,  $\beta$  sufficiently close to  $\beta(P)$ .

Now, since  $J$  vanishes at  $P$ , we have, by (2),  $\psi_*(P) = 0$ . By Taylor's theorem, since the second derivatives of  $\psi$  exist and are bounded,

$$(15) \quad \psi_* = (q - c_*)F_1 + (\theta - \theta(P))F_2$$

where  $F_1$  and  $F_2$  are bounded functions of  $q$  and  $\theta$ . Along the characteristic  $\alpha = \alpha(P)$ , we have, by (6)

$$(16) \quad \theta - \theta(P) = \int_{c_*}^q \sqrt{q^2/c^2 - 1} q^{-1} dq = m(q)(q - c_*)$$

where  $m$  is bounded. Hence, for  $\alpha = \alpha(P)$ ,  $\beta \geq \beta(P)$

$$(17) \quad \psi_* = (F_1 + mF_2)(q - c_*)$$

holds and therefore  $\left| \frac{\psi_*}{q - c_*} \right|$  is bounded. But then  $\left| \frac{\psi_*}{(q - c_*)^{1/4}} \right| \rightarrow 0$  for  $\alpha = \alpha(P)$ ,  $\beta \rightarrow \beta(P)$  and this contradicts (14). Thus  $J > 0$  for  $q = c_*$  as well as for  $q \geq c_*$  and the lemma is proved.

It remains only to apply the above lemma to the two situations: flow past a fixed airfoil and the construction of airfoils. We treat both these situations at once, and note first that the boundary of the domain of the flow in the hodograph plane is the streamline corresponding to the profile; since the sonic line intersects the profile twice, we need only show that  $J > 0$  on the profile to prove that  $J > 0$  for  $q > c_*$ .

The curvature  $K$  of any streamline is given by the formula

$$K = d\theta/ds = \theta_* \cos \theta + \theta_s \sin \theta.$$

But the transformation  $x = x(q, \theta)$ ,  $y = y(q, \theta)$  gives, if  $J$  does not vanish,

$$J\theta_* = y_*$$

$$J\theta_s = -x_*$$

and hence

$$\begin{aligned} KJq &= y,q \cos \theta - x,q \sin \theta \\ &= -\frac{1}{\rho} (y,\psi_s + x,\psi_s) = -\frac{i}{\rho} \psi_s. \end{aligned}$$

Using (2) we then find

$$(18) \quad K^2 J^2 q^2 = \frac{1}{\rho^2} \psi_s^2 \geq qJ$$

on the profile, for  $q \geq c_s \geq c$ .

Consider a set,  $M$ , of Mach numbers at infinity and a set of profiles, which depend continuously on  $M$ , and have finite curvature. Let  $K_m$  be the upper bound for the curvatures, and suppose that  $J > 0$  on each profile for  $M < M^*$ ;  $J = 0$  for  $M = M^*$  at some point on the supersonic section of the corresponding profile. Then, by (18)

$$J \geq 1/K_m^2 \hat{q} \quad \text{for} \quad M < M^*,$$

where  $\hat{q}$  is the limit speed. Therefore, since  $J$  is a continuous function of  $M$ ,

$$J \geq 1/K_m^2 \hat{q} > 0 \quad \text{for} \quad M = M^*;$$

i.e.,  $J$  cannot vanish on the boundary streamline. This fact is derived by Friedrichs in [3].

The lemma may now be applied to show that  $J > 0$  throughout the supersonic region and on the sonic line. We conclude therefore that a limiting line cannot appear in the flow about a fixed airfoil with a bounded supersonic region adjacent to it, and if it appears in the construction of profiles with adjacent supersonic regions then the curvature of the corresponding profile is infinite at some point.

### BIBLIOGRAPHY

- [1] Tsien, H. S., *The limiting line in mixed subsonic and supersonic flow of compressible fluids* Technical Note No. 961, National Advisory Committee for Aeronautics, 1944.
- [2] Nikolskii, A. A., and Taganov, G. I., *Flow of a gas in a local supersonic region and some conditions for the breakdown of potential flow*, Brown University Translation A9-T-17, 1948.
- [3] Friedrichs, K. O., *On the non-occurrence of a limiting line in transonic flow*, Communications on Pure and Applied Mathematics, Volume 1, No. 3, 1948, pp. 287-301.
- [4] Tsien, H. S., *Review of K. O. Friedrichs' article [3]*, Applied Mechanics Reviews, Volume 3, No. 753, 1950.
- [5] Maxwell, A. R., *A note on the hodograph transformation*, Quarterly of Applied Mathematics, Volume 10, No. 2, 1952, pp. 177-184.
- [6] Ringleb, F., *Exakte Lösungen der Differentialgleichungen einer adiabatischen Gasströmung*, Zeitschrift für angewandte Mathematik und Mechanik, Volume 20, 1940, pp. 185-198.
- [7] Tollmien, W., *Grenzlinien adiabatischer Potentialströmungen*, Zeitschrift für angewandte Mathematik und Mechanik, Volume 21, 1941, pp. 140-152.
- [8] Courant, R., and Friedrichs, K. O., *Supersonic Flow and Shock Waves*, Interscience, New York, 1948.